Global exponential stability of continuous-time interval neural networks

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This paper addresses global robust stability of a class of continuous-time interval neural networks that contain time-invariant uncertain parameters with their values being unknown but bounded in given compact sets. We first introduce the concept of diagonally constrained interval neural networks and present a necessary and sufficient condition for global exponential stability of these interval neural networks irregardless of any bounds of nondiagonal uncertain parameters in connection weight matrices. Then we extend the robust stability result to general interval neural networks by giving a sufficient condition. Simulation results illustrate the characteristics of the main results.

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I. INTRODUCTION

In recent years, many neural networks have been developed to solve various problems. In the design and hardware implementation of neural networks, a common problem is that parameters acquired in neural networks are inaccurate. To design neural networks, vital data such as the neuron firing rate and synaptic interconnection weights usually need to be measured, acquired, and processed by means of statistical estimation, which definitely leads to estimation errors. Moreover, parameter fluctuation in neural network circuits is also unavoidable. In practice, we can actually obtain the range of the vital data and the bounds of circuit parameters by engineering experience or from incomplete information. This fact implies that a good neural network should have certain robustness. Otherwise, the neural network may not be reliable in the practical applications. For example, when we apply an interval neural network having certain robustness property to solve optimization problems, we do not need to consider spurious suboptimal responses for each parameter value of the network, which is of great importance. Therefore, besides asymptotic stability of neural networks, which has been studied by many researchers (see, e.g., Refs. [1–12]), robust stability of neural networks has also received wide attention (e.g., Refs. [13–19]).

Generally speaking, there are two cases of concern on uncertain parameters. One case is that the bounds of uncertain parameters are constrained. For instance, Forti and Tesi [2] and Ye *et al.* [13] viewed the uncertain parameters as perturbations and gave some testable criteria for robust stability of continuous-time Hopfield neural networks. The conditions show that the matrix norm of the perturbations should be sufficiently small. Feng and Michel [15] established robust stability results for a class of discrete-time neural network model under small perturbations. In all these results, robustness means that the neural network is not overly sensitive to small perturbations. Recently, an *M*-matrix condition to guarantee robust stability for interval Hopfield neural networks was derived by Liao and Yu [14]. The other case of concern is that the bounds of uncertain parameters may be arbitrarily large. In Refs. [16–19], the absolute stability results, are related to robust stability results to some degree, indicate that the linear state self-feedback coefficient (diagonal) matrix of the network may be arbitrary negative definite when the connection weight matrix belongs to the particular set $\mathcal{I}_0$ or $\mathcal{M}_0$.

This paper first addresses the second case and derives a necessary and sufficient condition for global robust exponential stability of a class of continuous-time interval neural networks after introducing the concept of diagonally constrained interval networks. Then we extend the result to more general cases. The remainder of this paper is organized as follows. Section II describes some preliminaries. The main results are stated in Sec. III and IV. Illustrative results can be found in Sec. V. Finally, concluding remarks are made in Sec. VI.

II. PRELIMINARIES

Consider a typical continuous-time neural network model as follows:

$$\dot{x} = -Dx + Wg(x) + u, \quad x(0) = x_0, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ is the state vector, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in R^{n\times n}$ is a diagonal matrix with $d_i > 0, W = [w_{ij}] \in R^{n\times n}$ is a connection weight matrix, $u = (u_1, u_2, \ldots, u_n)^T \in R^n$ is an input vector, $g(x) = [g_1(x), g_2(x), \ldots, g_n(x)]^T$ is a vector-valued nonlinear activation function from R^n to R^n. In the following, let $\mathcal{GL}$ denote the class of *globally* Lipschitz continuous and monotone nondecreasing activation functions; that is, there exist $\overline{\ell}_i \geqslant \underline{\ell}_i \geqslant 0$ such that $\forall\ \theta, \rho \in R$ and $\theta \neq \rho$,

$$0 \leqslant \underline{\ell}_i \leqslant \frac{g_i(\theta) - g_i(\rho)}{\theta - \rho} \leqslant \overline{\ell}_i, \quad i = 1, 2, \ldots, n.$$

Definition 1 (Ref. [16]). An equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ of the neural network (1), which satisfies $-Dx^* + Wg(x^*) + u = 0$, is said to be globally asymptotically stable if it is locally stable in the sense of Lyapunov and globally attractive. The equilibrium x^* is said to be globally exponentially stable if there exist $\mu \geqslant 1$ and $\beta > 0$ such that $\forall\ x_0 \in R^n$, the positive half trajectory $x(t)$ of the neural network (1) satisfies

$$\|x(t)-x^*\| \leq \mu \|x_0 - x^*\| \exp(-\beta t), \quad t \geq 0.$$

Definition 2. Model (1) is a continuous-time interval neural network (CTINN) if D and W are time invariant and unknown but bounded in given compact sets; i.e., $0 < \underline{d}_i \leq d_i \leq \overline{d}_i, \underline{w}_{ij} \leq w_{ij} \leq \overline{w}_{ij}$.

D and W in Definition 2 are also called interval matrices. Let $\mathcal{D}_u$ and $\mathcal{W}_u$ be two prescribed compact sets to which D and W are confined, respectively. It should be noted that an interval network is actually a set of certain networks.

Definition 3. A CTINN (1) is called to be globally exponentially stable if it has a unique equilibrium x^* and x^* is globally exponentially stable for any given $u \in R^n$ and for any given parameters belonging to the prescribed given compact sets.

Definition 4. Let $g \in \mathcal{GL}$. The CTINN (1) is called a diagonally constrained CTINN (D,W,u) if $\overline{w}_{ii} < \underline{d}_i / \overline{\ell}_i, i = 1,2,\ldots,n$.

It is noted that for a diagonally constrained interval neural network (D,W,u) there is no restriction for nondiagonal entries of W. In other words, only the bounds of self-feedback terms in W are subject to constraints. Furthermore, the input vector u is arbitrary.

For a certain neural network, to study the global asymptotic stability of model (1) by applying the Lyapunov function method, we need to transform model (1) into a form where the origin is an equilibrium. Let x^* be an equilibrium of model (1) and $z = (z_1, z_2, \ldots, z_n)^T = x - x^*$ be a new state vector. Then, model (1) can be expressed in terms of z as

$$\dot{z} = -Dz + Wf(z), \quad z(0) = z_0, \tag{2}$$

where $f(z) = [f_1(z_1), \ldots, f_n(z_n)]^T = g(z+x^*) - g(x^*) \in \mathcal{GL}$ and $f(0)=0$. Hence, If a CTINN (1) has at least one equilibrium for each $D \in \mathcal{D}_u$ and $W \in \mathcal{W}_u$, then the robust stability of CTINN (1) is equivalent to that of the interval neural network (2).

In the sequel, for $x \in R^n$, let $\|x\|$ denote the Euclidean vector norm; i.e., $\|x\| = (x^T x)^{1/2}$. For a matrix $A \in R^{n \times n}$, let $\lambda_{\min}(A)$ [or $\lambda_{\max}(A)$] denote the smallest (or largest) eigenvalue of all the eigenvalues of A, and let $\|A\|$ denote the norm of A induced by the Euclidean vector norm; i.e., $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$. I_n is the $n \times n$ identity matrix. $[PA]^S = (A^T P + PA)/2$. Let $L = \mathrm{diag}(\ell_1, \ell_2, \ldots, \ell_n)$ and

$$M \triangleq W - D\overline{L}^{-1}, \tag{3}$$

where $\overline{L} = \mathrm{diag}(\overline{\ell}_1, \overline{\ell}_2, \ldots, \overline{\ell}_n)$.

Definition 5. An $n \times n$ matrix $C = [c_{ij}]$ is called to be a binary matrix if every entry c_{ij} of the matrix is either 0 or 1. Let Σ denote the set of all diagonally constrained interval neural networks (D,W,u) defined in Definition 4. Let Σ_c denote a subset of Σ determined by the following rule: $(D,W,u) \in \Sigma$ if $M = C * M$ where * represents the Hadamard product operation and $M = [m_{ij}]$ is defined in Eq. (3); i.e., $C * M = [c_{ij} m_{ij}]$. A binary matrix is also called a pattern matrix in Refs. [20,21].

Obviously, to check if a diagonally constrained interval network $(D,W,u) \in \Sigma_c$, we only need to check if $m_{ij} \equiv 0$ when $c_{ij} = 0$. According to Definition 5, the set Σ is divided into some subsets Σ_c by binary matrices C. For fixed order networks, since there are only a finite number of possible binary matrices C the number of subsets Σ_c is also finite. Note that some subsets Σ_c may not be disjoint. For example, if

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

then $\Sigma_{c_1} \subset \Sigma_{c_2} \subset \Sigma_{c_3}$.

In the following, it will be shown that there are special binary matrices C having the following property: every interval network $(D,W,u) \in \Sigma_c$ is globally exponentially stable.

Definition 6. Given an $n \times n$ binary matrix $C = [c_{ij}]$ (where $c_{ii} = 1, i = 1, \ldots, n$). If $\det C \equiv c_{11} c_{22} \cdots c_{nn} = 1$, then the binary matrix C is said to satisfy diagonal determinant condition.

Similarly, we also define the diagonal determinant condition for any matrix $M = [m_{ij}]_{n \times n}$ satisfying $\det M \equiv m_{11} m_{22} \cdots m_{nn} \neq 0$. For example, all possible 3×3 binary matrices satisfying diagonal determinant condition are shown below,

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & * & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{bmatrix}.$$

However, the following matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

does not satisfy diagonal determinant condition.

Lemma 1. Given any binary matrix $C = [c_{ij}]_{n \times n}$ where $c_{ii} = 1$ for all $i = 1, \ldots, n$. If there exist $c_{ki} = 1$ $(i \neq k)$ and $c_{jk} = 1$ $(j \neq k)$ for all $1 \leq k \leq n$, then $\det C \not\equiv c_{11} \cdots c_{nn}$.

Proof. Consider the worst case: there only exists a nonzero entry in the ith row and the ith column except for c_{ii} for all $i = 1, \ldots, n$. For convenience, these nonzero entries may be expressed by $c_{1i_1}, \ldots, c_{ni_n}$ where $i_j \neq j (j = 1, \ldots, n)$ and $i_k \neq i_s (k,s = 1, \ldots, n$, and $k \neq s)$. It is easy to see that there exists $c_{1i_1} \cdots c_{ni_n} = 1$ in $\det C$. Hence, $\det C \not\equiv c_{11} \cdots c_{nn}$.

When there are other nonzero entries in C besides those nonzero entries under the worst case above, it is trivial that $\det C \not\equiv c_{11} \cdots c_{nn}$.

Lemma 2. For any given binary matrix $C = [c_{ij}]_{n \times n} \triangleq [c_1^T \ \cdots \ c_n^T]^T$ where $n \geq 2$, if $\det C \equiv c_{11} \cdots c_{nn} = 1$, then there exists l $(1 \leq l \leq n)$ such that

$$c_l = (0, \ldots, 0,\ c_{ll},\ 0, \ldots, 0). \tag{4}$$

Proof. We prove Lemma 2 by mathematical induction. Consider a 2×2 matrix C. Since $\det C \equiv c_{11}c_{22}=1$, we immediately have either $c_{12}=0$ or $c_{21}=0$ and consequently, $c_1=(c_{11},\ 0)$ or $c_2=(0,\ c_{22})$.

Assume that for any $n\times n$ binary matrix C satisfying $\det C \equiv c_{11}\cdots c_{nn}=1$, there exists some $l\ (1\leqslant l\leqslant n)$ such that Eq. (4) holds. Then we will show that for any $(n+1)\times(n+1)$ binary matrix C^+ satisfying $\det C^+ \equiv c_{00}c_{11}\cdots c_{nn}=1$, there exists some $l^+\ (0\leqslant l^+\leqslant n)$ such that $c^+_{l^+}=(0,\ldots,0,c_{l^+l^+},0,\ldots,0)$ where C^+ is written by

$$C^+ \triangleq \begin{bmatrix} c_{00} & c_{01}\cdots c_{0n} \\ \vdots & C \\ c_{n0} & \end{bmatrix} \triangleq \begin{bmatrix} c_0^+ \\ \vdots \\ c_n^+ \end{bmatrix} \triangleq [\hat{c}_0^+\ \ \hat{c}_1^+ \cdots \hat{c}_n^+]. \tag{5}$$

First consider the first row and column of C^+. If $c_0^+ = (c_{00},0,\ldots,0)$, then it is trivial that $l^+=0$. If $\hat{c}_0^+ = (c_{00},0,\ldots,0)^T$, then $\det C^+ = c_{00}\det C = c_{00}c_{11}\cdots c_{nn}$. By assumption for C, there exists some $l\ (1\leqslant l\leqslant n)$ such that Eq. (4) holds. Hence, when $l^+=l$, $c^+_{l^+}=(0,\ c_l) = (0,\ldots,0,c_{ll},0,\ldots,0)$.

Now we only need to consider the following case: there exist $c^+_{0i}=1\ \ (i\neq 0)$ and $c^+_{j0}=1\ \ (j\neq 0)$.

Similarly, in turn for all $k=1,\ldots,n$ we only need to consider the following cases: there exist $c^+_{ki}=1\ \ (i\neq k)$ and $c^+_{jk}=1\ \ (j\neq k)$. Therefore, it follows from Lemma 1 that $\det C^+ \not\equiv c_{00}c_{11}\cdots c_{nn}$, which contradicts $\det C^+ \equiv c_{00}c_{11}\cdots c_{nn}$.

By mathematical induction, we have proved Lemma 2.

III. GLOBAL EXPONENTIAL STABILITY OF DIAGONALLY CONSTRAINED CTINNS

In this section, we state a global exponential stability condition for diagonally constrained CTINN (1). When $g\in\mathcal{GL}$ and $g(0)=0$, a necessary and sufficient condition for existence and uniqueness of equilibrium of Eq. (1) was given in Theorem 1 in Ref. [5]. In fact, $g(0)=0$ is not necessarily required. So,Theorem 1 in Ref. [5] can be restated as follows.

Lemma 3. Let $L=diag(\ell_1,\ell_2,\ldots,\ell_n)$ and $g\in\mathcal{GL}$. A certain neural network (1) has a unique equilibrium for any given $u\in R^n$ if and only if $-D+WL$ is nonsingular $\forall\ \ell_i \in[\underline{\ell}_i,\overline{\ell}_i]$.

Theorem 1. Let $g\in\mathcal{GL}$. Every diagonally constrained CTINN (D,W,u) in Σ_c is globally exponentially stable if and only if the matrix C satisfies diagonal determinant condition.

Proof (Sufficiency). Given any diagonally constrained interval network $(D,W,u)\in\Sigma_c$ where C is assumed to satisfy diagonal determinant condition. Then

$$\underline{d}_i > \overline{\ell}_i\overline{w}_{ii},\ \ i=1,2,\ldots,n, \tag{6}$$

and

$$\det(-D+WL)\equiv(-d_1+w_{11}\ell_1)\times(-d_2+w_{22}\ell_2)\cdots(-d_n+w_{nn}\ell_n). \tag{7}$$

Hence $-D+WL$ is nonsingular for any $D\in\mathcal{D}_u$, $W\in\mathcal{W}_u$, and any $\ell_i\in[\underline{\ell}_i,\overline{\ell}_i]$. Hence, from Lemma 3 it follows that any certain network (D,W,u) has an unique equilibrium where $D\in\mathcal{D}_u$ and $W\in\mathcal{W}_u$. Thus, we only need to discuss global exponential stability of the transformed interval network (2).

Next we first prove the following statement: there exists a positive diagonal matrix $P=\mathrm{diag}(p_1,p_2,\ldots,p_n)$ such that

$$(-D\overline{L}^{-1}+W)^T P + P(-D\overline{L}^{-1}+W)<0, \tag{8}$$

$\forall\ D\in\mathcal{D}_u$ and $\forall\ W\in\mathcal{W}_u$ if the interval network $(D,W,u)\in\Sigma_c$ and the binary matrix C satisfies diagonal determinant condition. Consider $n=1$. Since $(D,W,u)\in\Sigma_c$ and the binary matrix C satisfies diagonal determinant condition, we have $C=[1]$, $D=[d]$, $W=[w]$, and $\underline{d}>\overline{\ell}\overline{w}$. In this case, select $P=[1]$ that is positive definite, we can trivially guarantee that $(W-D/\overline{\ell})P+P(W-D/\overline{\ell})<0$ for any $D\in\mathcal{D}_u$ and $W\in\mathcal{W}_u$. Assume that the statement holds in the case of n. Then, we will show that the statement also holds in the case of $n+1$. Let the interval network $(D^+,W^+,u^+)\in\Sigma^+_{c^+}$ and the binary matrix C^+ satisfy diagonal determinant condition. Let

$$C^+=[c^+_{ij}]=\begin{bmatrix} c_1^+ \\ \vdots \\ c^+_{n+1}\end{bmatrix}.$$

Since C^+ satisfies diagonal determinant condition, by Lemma 2 there exists some $l^+(1\leqslant l^+\leqslant n+1)$ such that $c^+_{l^+} = (0,\ldots,0,c^+_{l^+l^+},0,\ldots,0)$ where $c^+_{l^+l^+}=1$.

Without a loss of generality, we assume $l^+=n+1$; i.e., $c^+_{n+1}=(0,\ldots,0,1)$. From $(D^+,W^+,u^+)\in\Sigma^+_{c^+}$, we may denote

$$W^+=\begin{bmatrix} & * \\ W & \vdots \\ & * \\ 0\cdots 0 & w_{n+1\ \ n+1}\end{bmatrix} \tag{9}$$

and $D^+=\mathrm{diag}(D,d_{n+1})$ where each $*$ denotes an uncertainty. We may also denote

$$C^+=\begin{bmatrix} & * \\ C & \vdots \\ & * \\ 0\cdots 0 & 1\end{bmatrix}, \tag{10}$$

where $*$ is 0 or 1. Then in view that $(D^+,W^+,u^+)\in\Sigma^+_{c^+}$ and C^+ satisfies diagonal determinant condition we can see that $(D,W,u)\in\Sigma_c$ and C above also satisfies diagonal-determinanted condition. By assumption, there exists a positive diagonal matrix P such that $(-D\overline{L}^{-1}+W)^T P + P(-D\overline{L}^{-1}+W)<0$ for any $D\in\mathcal{D}_u$ and $W\in\mathcal{W}_u$. Let $\overline{L^+}=\mathrm{diag}(\overline{L},\overline{\ell}_{n+1})$ and

$$P^+=\begin{bmatrix} P & 0\\ 0^T & p_{n+1}\end{bmatrix},$$

where 0 is a zero vector. Then

$$[-D^+(\overline{L^+})^{-1}+W^+]^T P^+ + P^+[-D^+(\overline{L^+})^{-1}+W^+]$$
$$=\begin{bmatrix} \pi(D,W,P) & m(W^+,P)\\ m(W^+,P)^T & 2p_{n+1}(-d_{n+1}\overline{\ell}_{n+1}^{-1}+w_{n+1\ n+1})\end{bmatrix}, \tag{11}$$

where $\pi(D,W,P)=(-D\overline{L}^{-1}+W)^T P+P(-D\overline{L}^{-1}+W)$ and $m(W^+,P)=P(*\cdots *)^T$. To guarantee that Eq. (11) is negative definite for all $D^+\in\mathcal{D}^+_{u^+}$ and $W^+\in\mathcal{W}^+_{u^+}$, we only need to choose

$$\max:\left\{\frac{\|m(W^+,P)\|^2}{(-d_{n+1}\overline{\ell}_{n+1}^{-1}+w_{n+1\ n+1})\lambda_{\max}[\pi(D,W,P)]}:\ D^+\in\mathcal{D}^+_{u^+},\ \ W^+\in\mathcal{W}^+_{u^+}\right\}. \tag{12}$$

This shows that the statement is true in the case $n+1$.

By the above mathematical induction, we have proved the statement.

Let $\underline{D}=\mathrm{diag}(\underline{d}_1,\underline{d}_2,\ldots,\underline{d}_n)$. Consider the following Lyapunov function

$$V(z)=\frac{1}{2}z^T\underline{D}^{-1}z+\frac{k}{\varepsilon}\sum_{i=1}^{n} p_i\int_0^{z_i} f_i(\rho)d\rho \tag{13}$$

with P as defined in Eq. (8), any fixed number $\varepsilon\in(0,1)$, and the constant k defined by

$$k\triangleq\frac{\max\limits_{W\in\mathcal{W}_u}(\|\underline{D}^{-1}W\|^2)}{4\min\limits_{D\in\mathcal{D}_u,W\in\mathcal{W}_u}(\lambda_{\min}\{([P(D\overline{L}^{-1}-W)]^S\})}\geqslant 0.$$

Computing the time derivative of $V(z)$ along the positive half trajectory of Eq. (2) yields

$$\begin{aligned}
\frac{dV(z)}{dt}&=[\underline{D}^{-1}z+(k/\varepsilon)Pf(z)]^T[-Dz+Wf(z)]\\
&=-z^T\underline{D}^{-1}Dz+z^T\underline{D}^{-1}Wf(z)(k/\varepsilon)f(z)^TPDz+(k/\varepsilon)f(z)^T[PW]^Sf(z)\\
&\leqslant -z^Tz+z^T\underline{D}^{-1}Wf(z)(k/\varepsilon)f(z)^TPD\overline{L}^{-1}f(z)+(k/\varepsilon)f(z)^T[PW]^Sf(z)\\
&=-z^Tz+z^T\underline{D}^{-1}Wf(z)-(k/\varepsilon)f(z)^T[P(D\overline{L}^{-1}-W)]^Sf(z)\\
&\leqslant -\|z\|^2+\left(\varepsilon\|z\|^2+\frac{1}{4\varepsilon}\|\underline{D}^{-1}Wf(z)\|^2\right)-(k/\varepsilon)f(z)^T[P(D\overline{L}^{-1}-W)]^Sf(z)\\
&\leqslant -\|z\|^2+\left(\varepsilon\|z\|^2+\frac{1}{4\varepsilon}\|\underline{D}^{-1}W\|^2\|f(z)\|^2\right)(k/\varepsilon)f(z)^T[P(D\overline{L}^{-1}-W)]^Sf(z)\\
&\leqslant -(1-\varepsilon)\|z\|^2+\left(\frac{1}{4\varepsilon}\max_{W\in\mathcal{W}_u}(\|\underline{D}^{-1}W\|^2)-(k/\varepsilon)\min_{D\in\mathcal{D}_u,W\in\mathcal{W}_u}(\lambda_{\min}\{[P(D\overline{L}^{-1}-W)]^S\})\right)\|f(z)\|^2\\
&=-(1-\varepsilon)\|z\|^2.
\end{aligned} \tag{14}$$

From Eq. (13) and $0\leqslant\rho f_i(\rho)\leqslant\overline{\ell}_i\rho^2$ $(\rho\in R)$ for $i=1,2,\ldots,n$ it follows that $\forall\, z\in R^n$,

$$\frac{1}{2}z^T\underline{D}^{-1}z\leqslant V(z)\leqslant\frac{1}{2}z^T\underline{D}^{-1}z+\frac{k}{\varepsilon}\sum_{i=1}^{n} p_i\overline{\ell}_i\int_0^{z_i}\rho d\rho$$
$$=\frac{1}{2}z^T\underline{D}^{-1}z+\frac{k}{2\varepsilon}\sum_{i=1}^{n} p_i\overline{\ell}_i z_i^2. \tag{15}$$

Let

$$\delta_1=\max_{1\leqslant i\leqslant n}(\underline{d}_i)\ \ \text{and}$$

$$\delta_2=\delta_2(\varepsilon)=\max_{1\leqslant i\leqslant n}\frac{1}{\underline{d}_i}+\frac{k}{\varepsilon}\sqrt{\sum_{i=1}^{n}(p_i\overline{\ell}_i)^2}. \tag{16}$$

Then

$$\|z\|^2/(2\delta_1)\leqslant V(z)\leqslant\delta_2\|z\|^2/2,\ \ \forall\, z\in R^n.$$

Thus, from Eq. (14) we have

$$dV(z)/dt \leqslant -\frac{2(1-\varepsilon)}{\delta_2}V(z), \quad \forall\, z\in R^n.$$

So,

$$V(z(t)) \leqslant V(z_0)\exp\left(-\frac{2(1-\varepsilon)}{\delta_2}t\right), \quad \forall\, z_0\in R^n, \quad \forall\, t\geqslant 0,$$

which yields

$$\|z(t)\| \leqslant \sqrt{\delta_1\delta_2}\|z_0\|\exp\left(-\frac{1-\varepsilon}{\delta_2}t\right), \quad \forall\, z_0\in R^n, \quad \forall\, t\geqslant 0.$$

Hence, model (2) is globally exponentially stable at the equilibrium $z=0$; that is, the diagonally constrained interval network (D,W,u) in Σ_c is globally exponentially stable at an exponential rate of at least $(1-\varepsilon)/\delta_2$ with $\delta_2=\delta_2(\varepsilon)$ defined in Eq. (16) where $\varepsilon\in(0,1)$ is any fixed number.

(Necessity). In order to prove the necessity of Theorem 1, we only need to equivalently prove the following. If a binary matrix $C=[c_{ij}]$ does not satisfy diagonal determinant condition, then, not every interval network $(D,W,u)\in\Sigma_c$ is globally exponentially stable, or there must be an interval network $(D,W,u)\in\Sigma_c$ that is not globally exponentially stable.

According to the definition of diagonal determinant condition, when C does not satisfy diagonal determinant condition; i.e., $\det C \not\equiv c_{11}\cdots c_{nn}=1$, then there must exist $c_{1i_1},c_{2i_2},\ldots,c_{ni_n}=1$ where $i_j(j=1,\ldots,n)\in\{1,2,\ldots,n\}$ and $\{c_{1i_1},c_{2i_2},\ldots,c_{ni_n}\}\cup\{c_{11},c_{22},\ldots,c_{nn}\}\neq\{c_{11},c_{22},\ldots,c_{nn}\}$.

Without a loss of generality, assume $c_{12}c_{21}c_{33}c_{44}\cdots c_{nn}=1$. Let $g\in\mathcal{G}L$ and $\overline{L}=I_n$. Now consider a diagonally constrained interval network (D,W,u) where

$$D=I_n, W=\begin{bmatrix} \frac{1}{2} & w_{12} & \cdots & 0 \\ w_{21} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} \end{bmatrix}, \qquad (17)$$

and entries w_{12} and w_{21} are interval entries. We easily check that the above interval network $(D,W,u)\in\Sigma_c$.

In the following, by contradiction we will prove that the network (D,W,u) is not globally asymptotically stable when $\overline{w}_{12}$ and $\overline{w}_{21}$ are sufficiently large and w_{12} and w_{21} are selected to be large enough.

Assume that the network (D,W,u) is globally asymptotically stable no matter how large w_{12} and w_{21} are. From Lemma 3 it follows that $-D+W\overline{L}=-D+W$ is nonsingular. On the other hand, rewrite

$$-D+W=\begin{bmatrix} -\frac{1}{2} & w_{12} & \cdots & 0 \\ w_{21} & -\frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2} \end{bmatrix}.$$

When $\overline{w}_{12}=\overline{w}_{21}=1/2$, we may select $w_{12}=w_{21}=1/2$. Then $\det(-D+W)=0$, which contradicts $-D+W$ being nonsingular.

For a diagonally constrained CTINN (D,W,u), if $-D\overline{L}^{-1}+W$ satisfies diagonal determinant condition, then based on Theorem 1 the CTINN (D,W,u) is globally exponentially stable, regardless of the bounds of nondiagonal entries of W for any given input vector u. Furthermore, since it is very easy to check if a matrix satisfies diagonal determinant condition, Theorem 1 gives a convenient way to ensure global exponential stability of an interval network. Theorem 1 is applicable for the diagonally constrained interval networks. However, it is invalid for general interval networks.

IV. GLOBAL EXPONENTIAL STABILITY OF GENERAL CTINNS

For general CTINNs (1), in this section we supply a result of global exponential stability. For a matrix satisfying diagonal-determinanted condition, there must exist some nondiagonal zero entries. Replace these zero entries by small perturbations. Now we consider the following general interval networks described by

$$\dot{x}=-Dx+(W+\Delta W)g(x)+u, \qquad (18)$$

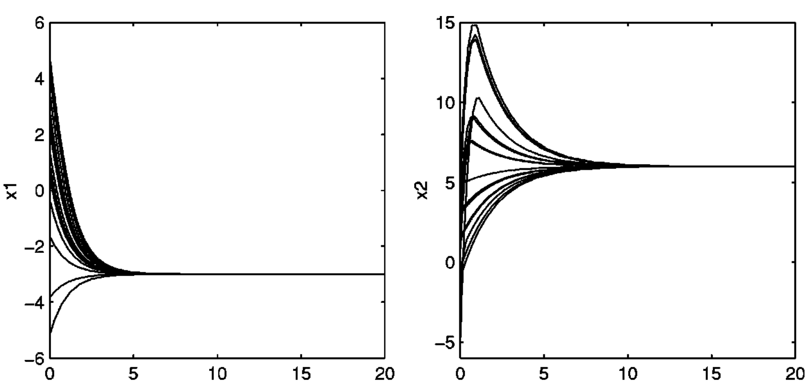

FIG. 1. Globally exponentially convergent transient states x_1 and x_2 in Example 1 with $w_{21}=6$ and $u=(-3,3)^T$.

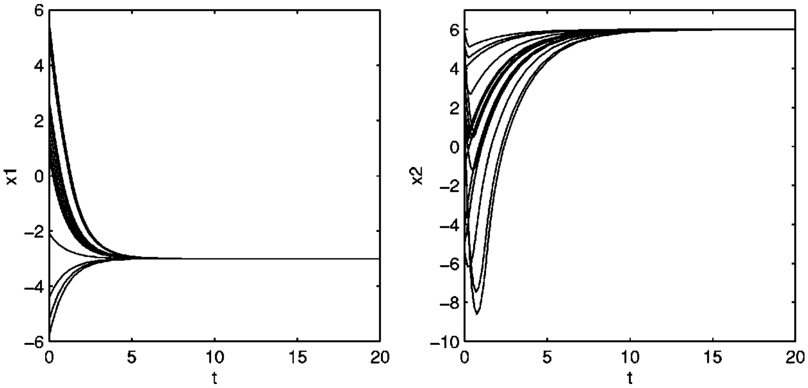

FIG. 2. Globally exponentially convergent transient states x_1 and x_2 in Example 1 with $w_{21}=-6$ and $u=(-3,3)^T$.

where $\Delta W=[\Delta w_{ij}]$, the interval network $(D,W,u)\in\Sigma_c$ and C satisfies diagonal determinant condition. Since the bounds of nondiagonal entries of W may be arbitrarily large in this case, for simplicity, we make the following assumption: for $i\neq j$, perturbation Δw_{ij} will be possibly encountered only when $w_{ij}\equiv 0$. Similar to the definitions of $\mathcal{D}_u$ and $\mathcal{W}_u$, let $\Delta\mathcal{W}_u$ be a prescribed compact set to which ΔW is confined. In the remaining part of this section, we will focus on the interval networks (18) with another uncertainties; i.e., small perturbations. Since the interval network $(D,W,u)\in\Sigma_c$ and C satisfies diagonal determinant condition, it follows from the proof of Theorem 1 that there exists a positive diagonal matrix $P=\mathrm{diag}(p_1,p_2,\ldots,p_n)>0$ such that

$$[P(-D\overline{L}^{-1}+W)]^S<0,\ \forall D\in\mathcal{D}_u \quad \text{and} \quad \forall W\in\mathcal{W}_u. \tag{19}$$

Next, we introduce the following result for general interval networks.

Theorem 2. The CTINN (18) is globally exponentially stable if

$$\xi\triangleq-\min_{D\in\mathcal{D}_u,W\in\mathcal{W}_u}(\lambda_{\min}([PM]^S))-p_{\max}\|\Delta W\|_u^*>0, \tag{20}$$

where M and P are defined in Eq. (3) and (19), respectively, $p_{\max}\triangleq\max\{p_1,p_2,\ldots,p_n\}$, and

$$\|\Delta W\|_u^*\triangleq\max_{\Delta W\in\Delta\mathcal{W}_u}\|\Delta W\|.$$

Proof. From Eq. (20) we have

$$\begin{aligned}[P(-D\overline{L}^{-1}+W+\Delta W)]^S &=[P\Delta W]^S+[P(-D\overline{L}^{-1}+W)]^S\\ &\leq[p_{\max}\|\Delta W\|_u^*-\min_{D\in\mathcal{D}_u,W\in\mathcal{W}_u}(\lambda_{\min}([PM]^S))]I_n.\\ &=-\xi I_n<0.\end{aligned} \tag{21}$$

Then (i) given any $0<L\leq\overline{L}$, we have $[P(W+\Delta W-DL^{-1})]^S\leq[P(W+\Delta W-D\overline{L}^{-1})]^S<0$, which shows that $W+\Delta W-DL^{-1}$ is stable or nonsingular, and consequently $-D+(W+\Delta W)L=(-DL^{-1}+W+\Delta W)L$ is nonsingular; (ii) given any $0\leq L\leq\overline{L}$ where there exists at least some $\ell_{i^*}=0$ $(1\leq i^*\leq n)$. Without a loss of generality, assume $0<\ell_i\leq\overline{\ell}_i, i=1,2,\cdots,n-1$, and $\ell_n=0$. Partition $D,L,W,\Delta W$ as

$$D=\begin{bmatrix}D_1 & 0\\ 0 & d_n\end{bmatrix},\quad L=\begin{bmatrix}L_1 & 0\\ 0 & \ell_n\end{bmatrix},\quad W=\begin{bmatrix}W_{11} & W_{1n}\\ W_{n1} & w_{nn}\end{bmatrix},$$

$$\Delta W=\begin{bmatrix}\Delta W_{11} & \Delta W_{1n}\\ \Delta W_{n1} & \Delta w_{nn}\end{bmatrix}.$$

Similar to case (i), we can deduce that $-D_1+(W_{11}+\Delta W_{11})L_1$ is nonsingular. So,

$$-D+(W+\Delta W)L=\begin{bmatrix}-D_1+(W_{11}+\Delta W_{11})L_1 & 0\\ (W_{n1}+\Delta W_{n1})L_1 & -d_n\end{bmatrix}$$

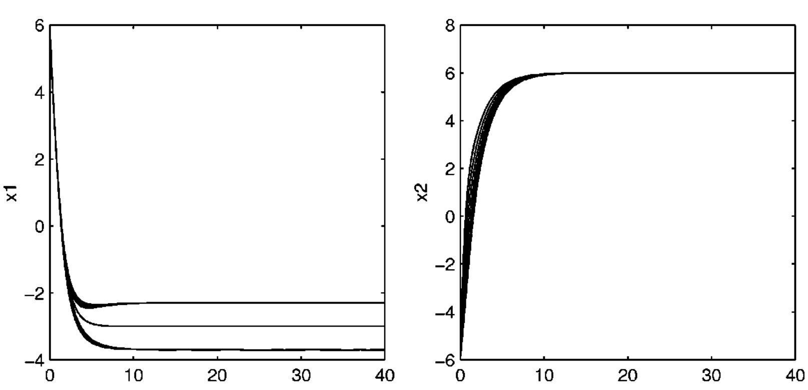

FIG. 3. Globally exponentially convergent transient states x_1 and x_2 in Example 2 with an input vector $u=(-3,3)^T$ and $x_0=(6,-6)^T$.

is nonsingular. In view of (i) and (ii), $-D+(W+\Delta W)L$ is nonsingular for any $D\in\mathcal{D}_u, W\in\mathcal{W}_u, \Delta W\in\Delta\mathcal{W}_u$, and $\forall \ell_i\in[\underline{\ell}_i,\overline{\ell}_i]$. Hence, given any $u\in R^n$, according to Lemma 3, any certain network (18) has a unique equilibrium x^*. So the robust stability of the interval network (18) is equivalent to that of the transformed interval network

$$\dot{z}=-Dz+(W+\Delta W)f(z), \tag{22}$$

where $f(z)=[f_1(z_1),f_2(z_2),\ldots,f_n(z_n)]^T=g(z+x^*)-g(x^*)\in\mathcal{G}L, f(0)=0$, and the equilibrium is the origin.

Consider the Lyapunov function $V(z)$ in Eq. (13) where the constant k is defined by

$$k\triangleq\frac{\max\limits_{W\in\mathcal{W}_u,\Delta W\in\Delta\mathcal{W}_u}(\|\underline{D}^{-1}(W+\Delta W)\|^2)}{4\xi}\geqslant 0.$$

Computing the time derivative of $V(z)$ along the positive half trajectory of Eq. (22) yields

$$\begin{aligned}
\frac{dV(z)}{dt}&=[\underline{D}^{-1}z+(k/\varepsilon)Pf(z)]^T[-Dz+(W+\Delta W)f(z)]\\
&=-z^T\underline{D}^{-1}Dz+z^T\underline{D}^{-1}(W+\Delta W)f(z)-(k/\varepsilon)f(z)^TPDz+(k/\varepsilon)f(z)^T[P(W+\Delta W)]^Sf(z)\\
&\leqslant -z^Tz+z^T\underline{D}^{-1}(W+\Delta W)f(z)-(k/\varepsilon)f(z)^TPD\overline{L}^{-1}f(z)+(k/\varepsilon)f(z)^T[P(W+\Delta W)]^Sf(z)\\
&=-z^Tz+z^T\underline{D}^{-1}(W+\Delta W)f(z)+(k/\varepsilon)f(z)^T[P(-D\overline{L}^{-1}+W+\Delta W)]^Sf(z)\\
&\leqslant -z^Tz+z^T\underline{D}^{-1}(W+\Delta W)f(z)-(k\xi/\varepsilon)\|f(z)\|^2 \qquad [\text{from Eq. (21)}]\\
&\leqslant -\|z\|^2+\left(\varepsilon\|z\|^2+\frac{1}{4\varepsilon}\|\underline{D}^{-1}(W+\Delta W)f(z)\|^2\right)-(k\xi/\varepsilon)\|f(z)\|^2\\
&\leqslant -\|z\|^2+\left(\varepsilon\|z\|^2+\frac{1}{4\varepsilon}\|\underline{D}^{-1}(W+\Delta W)\|^2\|f(z)\|^2\right)-(k\xi/\varepsilon)\|f(z)\|^2\leqslant-(1-\varepsilon)\|z\|^2\\
&\quad+\left(\frac{1}{4\varepsilon}\max_{W\in\mathcal{W}_u,\Delta W\in\Delta\mathcal{W}_u}(\|\underline{D}^{-1}(W+\Delta W)\|^2)-\frac{k\xi}{\varepsilon}\right)\|f(z)\|^2\\
&=-(1-\varepsilon)\|z\|^2.
\end{aligned}$$

Similar to the last part of the proof (sufficiency) of Theorem 1, we can see that the interval network (18) is globally exponentially stable.

For any given diagonally constrained CTINN (1) where W is replaced by $\widetilde{W}\triangleq[\widetilde{W}_{ij}]$, properly decompose CTINN (1) into the interval network (18). If the interval network $(D,W,u)\in\Sigma_c$ where C satisfies diagonal determinant condition, then we can obtain a positive diagonal matrix P as defined in Eq. (8) by Theorem 1. In this case, if the condition (20) further holds, then CTINN (1) is globally exponentially stable.

For a certain neural network (1), there exist many criteria for connection weight matrices to ascertain global asymptotic or exponential stability and absolute (or absolute

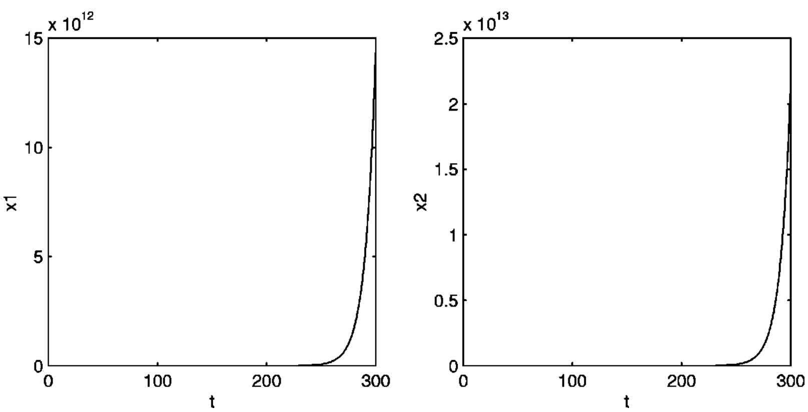

FIG. 4. Divergent transient states x_1 and x_2 in Example 2 with $w_{21}=0.9, \Delta w_{12}=0.41$, and $u=(-3,3)^T$.

exponential) stability (see, e.g., Refs. [17,18] for details). However, a characterization of any criteria (algebraic or otherwise) would be a hard problem for a general neural network (see Ref. [17]). In terms of computational complexity, the characterization may well be not polynomial hard. For an interval neural network (1) [including a set of certain neural networks (1)], Theorem 1 or Theorem 2 provides a simple and effective method to ascertain global exponential stability.

V. ILLUSTRATIVE EXAMPLES

Example 1. Consider a two-neuron diagonally constrained CTINN (1) where $g_i(\theta)=\max(\theta,0)$, $i=1,2$, $D=I_2$, $u\in R^2$,

$$W=\begin{bmatrix} w_{11} & 0 \\ w_{21} & w_{22} \end{bmatrix},$$

and $w_{11}=w_{22}=0.5$ are certain. We assume the uncertain parameter $|w_{21}|\leqslant\overline{w_{21}}$. Obviously, $\overline{L}=I_2$ and the matrix $M=-D\overline{L}^{-1}+W$ satisfies diagonal determinant condition. According to Theorem 1, the interval network is globally exponentially stable for any given input vector u no matter how large the bound $\overline{w_{21}}$. To verify this point by simulation, we use special selections of u and w_{21} (see Figs. 1 and 2). We choose 40 uniformly distributed random points in the set $[-6,6]\times[-6,6]$ as the initial states of the positive half trajectories of the neural network. It can be seen from Fig. 1 or Fig. 2 that all the trajectories from these initial points exponentially converge to a unique equilibrium. Figures 1 and 2 imply that this interval network is globally exponentially stable.

Example 2. Consider the CTINN (18) where $g_i(\theta)=\max(\theta,0)$, $i=1,2$, $D=I_2$, $u\in R^2$,

$$W=\begin{bmatrix} 0.5 & 0 \\ w_{21} & 0.5 \end{bmatrix},\quad \Delta W=\begin{bmatrix} 0 & \Delta w_{12} \\ 0 & 0 \end{bmatrix},\ \text{and}\quad |w_{21}|\leqslant 0.9.$$

As Example 1, we can see that the matrix $M=-D\overline{L}^{-1}+W$ satisfies the diagonal determinant condition. According to Theorem 1, the interval network (D,W,u) is globally exponentially stable. Moreover there exists a positive diagonal matrix $P=\mathrm{diag}(2,1)$ (consequently, $p_{\max}=2$) such that $[P(-D\overline{L}^{-1}+W)]^S<0$, $\forall\,|w_{21}|\leqslant 0.9$. A straightforward computation can obtain

$$\min_{|w_{21}|\leqslant 0.9}(\lambda_{\min}\{[P(D\overline{L}^{-1}-W)]^S\})=0.2352.$$

In view of Eq. (20), we easily get $|\Delta w_{12}|<0.1176$. Hence, when $|\Delta w_{12}|\leqslant 0.117$, Theorem 2 guarantees the interval network is globally exponentially stable. To simulate, we select w_{21} (varying from -0.9 and 0.9 by step length of 0.3) and Δw_{12} (varying from -0.117 and 0.117 by step length of 0.117). Figure 3 describes the transient states x_1 and x_2 with a given common input vector $u=(-3,\ 3)^T$ and the initial condition $x_0=(6,\ -6)^T$. Figure 3 shows the global exponential convergence of the states x_1 and x_2 of the interval network. When the bound of perturbation Δw_{12} is over large, Theorem 1 points out that the interval network is no longer globally exponentially stable. For example, when $w_{21}=0.9$, $\Delta w_{12}=0.41$, $u=(-3,\ 3)^T$, and the initial condition $x_0=(6,5)^T$, Fig. 4 shows the divergent transient states x_1 and x_2 of the network.

VI. CONCLUSIONS

In this paper, we study the global exponential stability of a class of continuous-time interval neural networks. Based on diagonally constrained interval neural network, we establish a necessary and sufficient condition for global exponential stability of these interval networks. The condition is easy to check. The bounds of nondiagonal uncertain parameters of the connection weight matrix may be arbitrarily large. We also extend the result to general interval networks. To demonstrate the characteristics of the derived results, two specific examples are discussed in detail.

ACKNOWLEDGMENT

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